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Normality of double fuzzy topological spaces

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ABSTRACT

In this paper, we introduce and characterize several types of normality in double fuzzy topological spaces. The effects of some types of functions on these types of normality are introduced.

1. Introduction

After the introduction of fuzzy sets [1], various mathematicians introduced generalizations of the notion of fuzzy set. Among others, Atanassov [2–7] introduced the notion of intuitionistic fuzzy set. Chang [8] used fuzzy sets to introduce the concept of a fuzzy topology. In 1985, Kubiak [9] and Šostak [10] introduced a new kind of fuzzy topological spaces depending on fuzzy sets. Later this concept was extended to intuitionistic fuzzy topological spaces by Çoker and his colleague [11–13]. In 1997, Samanta and Mondal [14,15] succeeded to make the topology itself intuitionistic. The resulting structure given the new name "intuitionistic gradation of openness".

The concept of fuzzy sets, intuitionistic fuzzy sets, fuzzy topology and intuitionistic fuzzy topology has been applied in different branches of science. For example, fuzzy topological spaces can be applied to solve many GIS problems. A suitable level cutting can induce a fuzzy topology. A traditional optimal thresholding induced a fuzzy topology, which results in minimum error segmentation. This fuzzy topology also gives a decomposition of the object of an image, which includes interior, boundary and exterior. Then, a further treatment is operating on the boundary by the concept of connectivity in fuzzy topology (see [16,17]).

In this paper, we follow the suggestion of Garcia and Rodabaugh [18] that double fuzzy set is a more appropriate name than intuitionistic fuzzy set, and therefore adopt the term double gradation fuzzy topology (double fuzzy topology, for simplification) for the intuitionistic gradation of openness of Samanta and Mondal.

In this paper, we introduce the concepts of double fuzzy almost normal, double fuzzy normal, double fuzzy mildly normal spaces in double fuzzy topological spaces and then we investigate some of their characteristic properties.

2. Preliminaries

Throughout this paper, let X be a nonempty set and I is the closed unit interval $[0, 1]$. $I_\circ = (0, 1]$ and $I_1 = [0, 1)$. The family of all fuzzy sets on X is denoted by I^X. By $\underline{0}$ and $\underline{1}$, we denote the smallest and the greatest fuzzy sets on X. For a fuzzy set $\lambda \in I^X$, $\underline{1} - \lambda$ denotes its complement. Given a function $f : X \to Y$, $f(\lambda)$ and $f^{-1}(\lambda)$ define the direct image and

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the inverse image of f, defined by $f(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ and $f^{-1}(\nu)(x) = \nu(f(x))$, for each $\lambda \in I^X$, $\nu \in I^Y$ and $x \in X$, respectively. For fuzzy sets λ and μ in X, we write $\lambda q \mu$ to mean that λ is quasi coincident with μ, that is, there exists at least one point $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Negation of such a statement is denoted as $\lambda \bar{q} \mu$. Notions and notations not described in this paper are standard and usual.

Definition 2.1 ([18,14,15]). The pair of functions $\mathcal{T}, \mathcal{T}^* : I^X \to I$ is called a double fuzzy topology on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\lambda) \le 1 - \mathcal{T}^*(\lambda)$ for each $\lambda \in I^X$.
(O2) $\mathcal{T}(\lambda_1 \wedge \lambda_2) \ge \mathcal{T}(\lambda_1) \wedge \mathcal{T}(\lambda_2)$ and $\mathcal{T}^*(\lambda_1 \wedge \lambda_2) \le \mathcal{T}^*(\lambda_1) \vee \mathcal{T}^*(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.
(O3) $\mathcal{T}(\bigvee_{i\in\Gamma} \lambda_i) \ge \bigwedge_{i\in\Gamma} \mathcal{T}(\lambda_i)$ and $\mathcal{T}^*(\bigvee_{i\in\Gamma} \lambda_i) \le \bigvee_{i\in\Gamma} \mathcal{T}^*(\lambda_i)$ for any $\{\lambda_i\}_{i\in\Gamma} \subset I^X$.

The triplet $(X, \mathcal{T}, \mathcal{T}^*)$ is called a double fuzzy topological space (dfts, for short). $\mathcal{T}(\lambda)$ and $\mathcal{T}^*(\lambda)$ may be interpreted as a gradation of openness and gradation of non-openness for λ. A function $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is said to be double fuzzy continuous if $\mathcal{T}_1(f^{-1}(\nu)) \ge \mathcal{T}_2(\nu)$ and $\mathcal{T}_1^*(f^{-1}(\nu)) \le \mathcal{T}_2^*(\nu)$ for each $\nu \in I^Y$.

Theorem 2.2 ([13,19]). *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a dfts. Then for each $r \in I_0$, $s \in I_1$ and $\lambda \in I^X$, we define an operator $C_{\mathcal{T},\mathcal{T}^*} : I^X \times I_0 \times I_1 \to I^X$ as follows:*

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \bigwedge\{\mu \in I^X | \lambda \le \mu, \mathcal{T}(\underline{1} - \mu) \ge r, \mathcal{T}^*(\underline{1} - \mu) \le s\}.$$

For $\lambda, \mu \in I^X$, $r, r_1, r_2 \in I_0$ and $s, s_1, s_2 \in I_1$, the operator $C_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(C1) $C_{\mathcal{T},\mathcal{T}^*}(\underline{0}, r, s) = \underline{0}$.
(C2) $\lambda \le C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$.
(C3) $C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \vee C_{\mathcal{T},\mathcal{T}^*}(\mu, r, s) = C_{\mathcal{T},\mathcal{T}^*}(\lambda \vee \mu, r, s)$.
(C4) $C_{\mathcal{T},\mathcal{T}^*}(\lambda, r_1, s_1) \le C_{\mathcal{T},\mathcal{T}^*}(\lambda, r_2, s_2)$ *if* $r_1 \le r_2$ *and* $s_1 \ge s_2$.
(C5) $C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s), r, s) = C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$.

Theorem 2.3 ([13,19]). *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a dfts. Then, for each $r \in I_1$, $s \in I_0$ and $\lambda \in I^X$, we define an operator $I_{\mathcal{T},\mathcal{T}^*} : I^X \times I_1 \times I_0 \to I^X$ as follows:*

$$I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \bigvee\{\mu \in I^X | \mu \le \lambda, \mathcal{T}(\mu) \ge r, \mathcal{T}^*(\mu) \le s\}.$$

For $\lambda, \mu \in I^X$, $r, r_1, r_2 \in I_1$ and $s, s_1, s_2 \in I_0$, the operator $I_{\mathcal{T},\mathcal{T}^*}$ satisfies the following statements:

(I1) $I_{\mathcal{T},\mathcal{T}^*}(\underline{1} - \lambda, r, s) = \underline{1} - C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$.
(I2) $I_{\mathcal{T},\mathcal{T}^*}(\underline{1}, r, s) = \underline{1}$.
(I3) $I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \le \lambda$.
(I4) $I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \wedge I_{\mathcal{T},\mathcal{T}^*}(\mu, r, s) = I_{\mathcal{T},\mathcal{T}^*}(\lambda \wedge \mu, r, s)$.
(I5) $I_{\mathcal{T},\mathcal{T}^*}(\lambda, r_1, s_1) \ge I_{\mathcal{T},\mathcal{T}^*}(\lambda, r_2, s_2)$ *if* $r_1 \le r_2$ *and* $s_1 \ge s_2$.
(I6) $I_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s), r, s) = I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$.
(I7) *If* $I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s), r, s) = \lambda$, *then* $C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\underline{1} - \lambda, r, s), r, s) = \underline{1} - \lambda$.

Definition 2.4. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a dfts, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$.

(1) A fuzzy set λ is called (r, s)-regular fuzzy open (for short, (r, s)-rfo) if

$$\lambda = I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s), r, s).$$

(2) A fuzzy set λ is called (r, s)-regular fuzzy closed (for short, (r, s)-rfc) if

$$\lambda = C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s), r, s).$$

Definition 2.5. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a dfts, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. A fuzzy set λ is called (r, s)-fuzzy α-open (for short, (r, s)-fαo) if

$$\lambda \le I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s), r, s), r, s).$$

λ is called (r, s)-fuzzy α-closed (for short, (r, s)-fαc) if and only if $\underline{1} - \lambda$ is (r, s)-fαo. The (r, s)-fuzzy α-closure and (r, s)-fuzzy α-interior of λ is defined by

$$\alpha C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \bigwedge\{\mu \in I^X | \lambda \le \mu, \mu \text{ is } (r, s) - f\alpha c\},$$
$$\alpha I_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \bigvee\{\mu \in I^X | \mu \le \lambda, \mu \text{ is } (r, s) - f\alpha o\}.$$

Definition 2.6. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a dfts, $\lambda, \mu \in I^X$, $r \in I_0$ and $s \in I_1$.

(1) A fuzzy set λ is called (r, s)-generalized fuzzy closed [20] (for short, (r, s)-gfc) if $C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \leq \mu$ whenever $\lambda \leq \mu$, $\mathcal{T}(\mu) \geq r$ and $\mathcal{T}^*(\mu) \leq s$. λ is called (r, s)-generalized fuzzy open (for short, (r, s)-gfo) if $\underline{1} - \lambda$ is (r, s)-gfc set.
(2) A fuzzy set λ is called (r, s)-regular generalized fuzzy closed (for short, (r, s)-rgfc) if $C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \leq \mu$ whenever $\lambda \leq \mu$ and μ is (r, s)-rfo. λ is called (r, s)-regular generalized fuzzy open (for short, (r, s)-rgfo) iff $\underline{1} - \lambda$ is (r, s)-rgfc set.
(3) A fuzzy set λ is called (r, s)-generalized fuzzy α-closed (for short, (r, s)-gfαc) if $\alpha C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \leq \mu$ whenever $\lambda \leq \mu$, $\mathcal{T}(\mu) \geq r$ and $\mathcal{T}^*(\mu) \leq s$. λ is called (r, s)-generalized fuzzy α-open (for short, (r, s)-gfαo) iff $\underline{1}-\lambda$ is (r, s)-gfαc set.
(4) A fuzzy set λ is called (r, s)-regular generalized fuzzy α-closed (for short, (r, s)-rgfαc) if $\alpha C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \leq \mu$ whenever $\lambda \leq \mu$ and μ is (r, s)-rfo. λ is called (r, s)-regular generalized fuzzy α-open (for short, (r, s)-rgfαo) iff $\underline{1}-\lambda$ is (r, s)-rgfαc set.

Remark 2.7. From the above definitions, it is not difficult to conclude that the following diagram of implications is true:

$$\begin{array}{ccc} (r,s)\text{-gfc } \textit{set} & \Rightarrow & (r,s)\text{-rgfc } \textit{set} \\ \Downarrow & & \Downarrow \\ (r,s)\text{-gf}\alpha\text{c } \textit{set} & \Rightarrow & (r,s)\text{-rgf}\alpha\text{c } \textit{set} \end{array}$$

Example 2.1. Let $X = \{a, b\}$ and let λ_1, μ_1 and ν_1 are fuzzy sets defined by

$$\lambda_1(a) = 0.2, \qquad \lambda_1(b) = 0.4;$$
$$\mu_1(a) = 0.9, \qquad \mu_1(b) = 0.4;$$
$$\nu_1(a) = 0.1, \qquad \nu_1(b) = 0.4.$$

Define $(\mathcal{T}_1, \mathcal{T}_1^*)$ and $(\mathcal{T}_2, \mathcal{T}_2^*)$ on X as follows:

$$\mathcal{T}_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \lambda_1; \\ \frac{1}{3}, & \text{if } \lambda = \mu_1; \\ 0, & \text{otherwise}. \end{cases}, \qquad \mathcal{T}_1^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \lambda_1; \\ \frac{1}{3}, & \text{if } \lambda = \mu_1; \\ 1, & \text{otherwise}. \end{cases}$$

$$\mathcal{T}_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{2} & \text{if } \lambda = \mu_1; \\ 0 & \text{otherwise}, \end{cases}, \qquad \mathcal{T}_2^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{2} & \text{if } \lambda = \mu_1; \\ 1 & \text{otherwise}. \end{cases}$$

(a) In $(\mathcal{T}_1, \mathcal{T}_1^*)$, it can be shown that ν_1 is an (r, s)-gfαc set, but it is not an (r, s)-gfc set. Also, ν_1 is an (r, s)-rgfαc set, but it is not an (r, s)-rgfc set.
(b) In $(\mathcal{T}_2, \mathcal{T}_2^*)$, ν_1 is an (r, s)-rgfαc set, but it is not an (r, s)-gfαc set. Also, ν_1 is an (r, s)-rgfc set, but it is not an (r, s)-gfc set.

Definition 2.8. Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be a function between dfts's $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$. Then, the function f is called:

(1) double fuzzy regular continuous if $f^{-1}(\nu)$ is (r, s)-rfo, for each $\nu \in I^Y$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}_2(\nu) \geq r$ and $\mathcal{T}_2^*(\nu) \leq s$.
(2) double fuzzy regular irresolute if $f^{-1}(\nu)$ is (r, s)-rfo, for each $\nu \in I^Y$, $r \in I_0$ and $s \in I_1$ such that ν is (r, s)-rfo set.
(3) double fuzzy almost regular generalized continuous if $f^{-1}(\nu)$ is (r, s)-rgfo, for each $\nu \in I^Y$, $r \in I_0$ and $s \in I_1$ such that ν is (r, s)-rfo set.
(4) double fuzzy almost continuous if $\mathcal{T}_1(f^{-1}(\nu)) \geq r$ and $\mathcal{T}_1^*(f^{-1}(\nu)) \leq s$, for each $\nu \in I^Y$, $r \in I_0$ and $s \in I_1$ such that ν is (r, s)-rfo set.

Definition 2.9. Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be a function between dfts's $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$. Then, the function f is called:

(1) double fuzzy regular closed if $f(\lambda)$ is (r, s)-rfc, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}_1(\underline{1}-\lambda) \geq r$ and $\mathcal{T}_1^*(\underline{1}-\lambda) \leq s$.
(2) double fuzzy almost closed if $\mathcal{T}_2(\underline{1} - f(\lambda)) \geq r$ and $\mathcal{T}_2^*(\underline{1} - f(\lambda)) \leq s$, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that λ is (r, s)-rfc set.
(3) double fuzzy almost generalized closed if $f(\lambda)$ is (r, s)-gfc, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that λ is (r, s)-rfc set.
(4) double fuzzy almost regular generalized closed if $f(\lambda)$ is (r, s)-rgfc, for each $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$ such that λ is (r, s)-rfc set.

3. Double fuzzy normal spaces

Definition 3.1. A dfts $(X, \mathcal{T}, \mathcal{T}^*)$ is said to be:

(1) double fuzzy almost normal space if for each $\lambda_1, \lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r$, $\mathcal{T}^*(\underline{1}-\lambda_1) \leq s$, λ_2 is (r, s)-rfc set and $\lambda_1 \bar{q} \lambda_2$, there exist $\mu_1, \mu_2 \in I^X$ such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ with $\mu_1 \bar{q} \mu_2$.
(2) double fuzzy normal space if for each $\lambda_1, \lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r$, $\mathcal{T}^*(\underline{1}-\lambda_1) \leq s$, $\mathcal{T}(\underline{1}-\lambda_2) \geq r$, $\mathcal{T}^*(\underline{1}-\lambda_2) \leq s$ and $\lambda_1 \bar{q} \lambda_2$, there exists $\mu_1, \mu_2 \in I^X$ such that $\mathcal{T}(\mu_1) \geq r$, $\mathcal{T}^*(\mu_1) \leq s$, $\mathcal{T}(\mu_2) \geq r$, $\mathcal{T}^*(\mu_2) \leq s$, $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.
(3) double fuzzy mildly normal space if for each $\lambda_1, \lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that λ_1 and λ_2 are (r, s)-rfc sets and $\lambda_1 \bar{q} \lambda_2$, there exists $\mu_1, \mu_2 \in I^X$ such that $\mathcal{T}(\mu_1) \geq r$, $\mathcal{T}^*(\mu_1) \leq s$, $\mathcal{T}(\mu_2) \geq r$, $\mathcal{T}^*(\mu_2) \leq s$, $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.

Clearly, every double fuzzy normal space is double fuzzy almost normal space and every double fuzzy almost normal space is double fuzzy mildly normal space, but the converse need not be true in general as the following example shows.

Example 3.1. Let $X = \{x\}$ and define the fuzzy sets μ, ν, ρ, γ and ω as follows:

$$\mu(x) = 0.4, \qquad \nu(x) = 0.7, \qquad \rho(x) = 0.8, \qquad \gamma(x) = 0.6, \qquad \omega(x) = 0.2.$$

Define the double fuzzy topologies as follows:

$$\mathcal{T}_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \mu; \\ \frac{1}{3}, & \text{if } \lambda = \nu; \\ 0, & \text{otherwise.} \end{cases}, \qquad \mathcal{T}_1^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \mu; \\ \frac{1}{3}, & \text{if } \lambda = \nu; \\ 1, & \text{otherwise.} \end{cases}$$

$$\mathcal{T}_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \nu; \\ \frac{1}{2}, & \text{if } \lambda = \rho; \\ \frac{1}{5}, & \text{if } \lambda = \gamma; \\ \frac{1}{6}, & \text{if } \lambda = \omega; \\ 0, & \text{otherwise.} \end{cases}, \qquad \mathcal{T}_2^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \underline{0}, \underline{1}; \\ \frac{1}{4}, & \text{if } \lambda = \nu; \\ \frac{1}{2}, & \text{if } \lambda = \rho; \\ \frac{1}{5}, & \text{if } \lambda = \gamma; \\ \frac{1}{6}, & \text{if } \lambda = \omega; \\ 1, & \text{otherwise.} \end{cases}$$

(1) $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal space but not double fuzzy almost normal space.
(2) $(X, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy almost normal space but not double fuzzy normal space.

Theorem 3.2. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be a dfts. Then, the following statements are equivalent:*

(1) *$(X, \mathcal{T}, \mathcal{T}^*)$ is a double fuzzy mildly normal space.*
(2) *For each pair of (r, s)-rfc sets $\lambda_1, \lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\lambda_1 \bar{q} \lambda_2$, there exist (r, s)-rfo sets μ_1, μ_2 such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.*
(3) *For each pair of (r, s)-rfc sets $\lambda_1, \lambda_2 \in I^X$, $r \in I_0$ and $s \in I_0$ such that $\lambda_1 \bar{q} \lambda_2$, there exist (r, s)-gfo sets μ_1, μ_2 such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.*
(4) *For each pair of (r, s)-rfc sets $\lambda_1, \lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\lambda_1 \bar{q} \lambda_2$, there exist (r, s)-gfαo sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.*
(5) *For each (r, s)-rfc set $\lambda \in I^X$ and each (r, s)-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists $\rho \in I^X$ such that $\mathcal{T}(\rho) \geq r$, $\mathcal{T}^*(\rho) \leq s$ and*

$$\lambda \leq \rho \leq C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s) \leq \mu.$$

(6) *For each (r, s)-rfc set $\lambda \in I^X$ and each (r, s)-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an (r, s)-rfo set $\rho \in I^X$ such that*

$$\lambda \leq \rho \leq C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s) \leq \mu.$$

(7) *For each (r, s)-rfc set $\lambda \in I^X$ and each (r, s)-rfo set $\mu \in I^X$ such that $\lambda \leq \mu$, there exists an (r, s)-gfo set $\rho \in I^X$ such that*

$$\lambda \leq \rho \leq C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s) \leq \mu.$$

(8) *For each* (r,s)*-rfc set* $\lambda \in I^X$ *and each* (r,s)*-rfo set* $\mu \in I^X$ *such that* $\lambda \le \mu$*, there exists an* (r,s)-gfαo *set* $\rho \in I^X$ *such that*

$$\lambda \le \rho \le \alpha C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \le \mu.$$

(9) *For each* (r,s)*-rfc set* $\lambda \in I^X$ *and each* (r,s)*-rfo set* $\mu \in I^X$ *such that* $\lambda \le \mu$*, there exists an* (r,s)-fαo *set* $\rho \in I^X$ *such that*

$$\lambda \le \rho \le \alpha C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \le \mu.$$

(10) *For each* (r,s)*-rfc sets* $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ *and* $s \in I_1$ *such that* $\lambda_1\bar{q}\lambda_2$*, there exist* (r,s)-fαo *sets* μ_1 *and* μ_2*, such that* $\lambda_1 \le \mu_1$, $\lambda_2 \le \mu_2$ *and* $\mu_1\bar{q}\mu_2$.

Proof. (1) $\Rightarrow$ (5). For each $r \in I_0$ and $s \in I_1$, let λ be (r,s)-rfc set and let μ be an (r,s)-rfo set such that $\lambda \le \mu$. Then, $\lambda\bar{q}\underline{1}-\mu$. By (1), there exist (r,s)-rfo sets λ_1 and μ_1, such that $\lambda \le \lambda_1$, $\underline{1}-\mu \le \lambda_1$ and $\lambda_1\bar{q}\mu_1$. Thus

$$\lambda \le \lambda_1 \le C_{\mathcal{T},\mathcal{T}^*}(\lambda_1,r,s) \le \underline{1}-\mu_1 \le \mu.$$

(5) $\Rightarrow$ (2). Let λ_1 and λ_2 are (r,s)-rfc sets such that $\lambda_1\bar{q}\lambda_2$. Then, $\lambda_1 \le \underline{1}-\lambda_2$. By (5), there exists $\rho \in I^X$ such that $\mathcal{T}(\rho) \ge r$, $\mathcal{T}^*(\rho) \le s$ and

$$\lambda_1 \le \rho \le C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \le \underline{1}-\lambda_2.$$

It follows that

$$\lambda_1 \le I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s),r,s) \le \underline{1}-\lambda_2$$

and

$$\lambda_2 \le \underline{1}-C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) = I_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,r,s).$$

Then, $\mu_1 = I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s),r,s)$ and $\mu_2 = I_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\rho,r,s)$ are (r,s)-rfo sets such that $\lambda_1 \le \mu_1$, $\lambda_2 \le \mu_2$ and $\mu_1\bar{q}\mu_2$.
(2) $\Rightarrow$ (6) Let λ_1 be any (r,s)-rfc set and let λ_2 be (r,s)-rfo set such that $\lambda_1 \le \lambda_2$. Then, $\lambda_1\bar{q}\underline{1}-\lambda_2$. According to (2), there exist (r,s)-rfo sets μ_1 and μ_2 such that $\lambda_1 \le \mu_1$, $\underline{1}-\lambda_2 \le \mu_2$ and $\mu_1\bar{q}\mu_2$. Let $\rho = I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s),r,s)$. Then, ρ is an (r,s)-rfo set such that

$$\lambda_1 \le \rho \le C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \le \underline{1}-\mu_2 \le \lambda_2.$$

(6) $\Rightarrow$ (2) Let λ_1 and λ_2 be (r,s)-rfc sets such that $\lambda_1\bar{q}\lambda_2$. Then, $\lambda_1 \le \underline{1}-\lambda_2$. By (6), there exists (r,s)-rfo set ρ such that

$$\lambda_1 \le \rho \le C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \le \underline{1}-\lambda_2.$$

For $C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \le \underline{1}-\lambda_2$, there exists (r,s)-rfo set μ such that

$$C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \le \mu \le C_{\mathcal{T},\mathcal{T}^*}(\mu,r,s) \le \underline{1}-\lambda_2.$$

Then, $\lambda_2 \le \underline{1}-C_{\mathcal{T},\mathcal{T}^*}(\mu,r,s) = I_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\mu,r,s)$, $I_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\mu,r,s)$ is (r,s)-rfo set and $\rho\bar{q}I_{\mathcal{T},\mathcal{T}^*}(\underline{1}-\mu,r,s)$.
(4) $\Rightarrow$ (8) Let λ_1 be an (r,s)-rfc set and let λ_2 be (r,s)-rfo set such that $\lambda_1 \le \lambda_2$. Then, $\lambda_1\bar{q}\underline{1}-\lambda_2$. By (4), there exist (r,s)-gfαo sets μ_1 and μ_2 such that $\lambda_1 \le \mu_1$, $\underline{1}-\lambda_2 \le \mu_2$ and $\mu_1\bar{q}\mu_2$. Since μ_2 is (r,s)-gfαo set and $\underline{1}-\lambda_2$ is (r,s)-rfc set, from $\underline{1}-\lambda_2 \le \mu_2$ follows that $\underline{1}-\lambda_2 \le \alpha I_{\mathcal{T},\mathcal{T}^*}(\mu_2,r,s)$. Thus

$$\underline{1}-\lambda_2 \le \alpha I_{\mathcal{T},\mathcal{T}^*}(\mu_2,r,s) \le \mu_2 \le \underline{1}-\mu_1.$$

Since $\underline{1}-I_{\mathcal{T},\mathcal{T}^*}(\mu_2,r,s)$ is an (r,s)-gfαc set, from $\mu_1 \le \underline{1}-\alpha I_{\mathcal{T},\mathcal{T}^*}(\mu_2,r,s)$, we obtain that

$$\alpha C_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s) \le \underline{1}-\alpha I_{\mathcal{T},\mathcal{T}^*}(\mu_2,r,s).$$

Hence

$$\lambda_1 \le \mu_1 \le \alpha C_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s) \le \lambda_2.$$

(8) $\Rightarrow$ (9) Let λ_1 be an (r,s)-rfc set and let λ_2 be an (r,s)-rfo set such that $\lambda_1 \le \lambda_2$. Then, $\lambda_1\bar{q}\underline{1}-\mu$. By (8), there exists an (r,s)-gfαo set μ_1 such that

$$\lambda_1 \le \mu_1 \le \alpha C_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s) \le \lambda_2.$$

Since μ_1 is an (r,s)-gfαo set, from $\lambda_1 \le \mu_1$ follows that $\lambda_1 \le \alpha I_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s)$. Then, $\mu_2 = \alpha I_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s)$ is an (r,s)-fαo set and

$$\lambda_1 \le \mu_2 \le \alpha C_{\mathcal{T},\mathcal{T}^*}(\mu_2,r,s) \le \alpha C_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s) \le \lambda_2.$$

(9) $\Rightarrow$ (10) Let λ_1 and λ_2 are (r,s)-rfc sets such that $\lambda_1\bar{q}\lambda_2$. Then, $\lambda_1 \le \underline{1}-\lambda_2$. By (9), there exists (r,s)-fαo set μ_1 such that

$$\lambda_1 \le \mu_1 \le \alpha C_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s) \le \underline{1}-\lambda_2.$$

Then, $\mu_2 = \underline{1}-\alpha C_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s)$ is an (r,s)-fαo set and $\mu_1\bar{q}\mu_2$.

(10) $\Rightarrow$ (1) Let λ_1 and λ_2 are (r,s)-rfc sets such that $\lambda_1\bar{q}\lambda_2$. Then, $\lambda_1 \leq \underline{1} - \lambda_2$. By (10), there exist (r,s)-fαo sets μ_1 and μ_2 such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1\bar{q}\mu_2$. Let $\rho_1 = I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mu_1,r,s),r,s),r,s)$ and $\rho_2 = I_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mu_2,r,s),r,s),r,s)$. Then, $\mathcal{T}(\rho_1) \geq r, \mathcal{T}^*(\rho_1) \leq s$ and $\mathcal{T}(\rho_2) \geq r, \mathcal{T}^*(\rho_1) \leq s$ and $\rho_1\bar{q}\rho_2$. Hence $(X,\mathcal{T},\mathcal{T}^*)$ is double fuzzy normal space.
The implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4), (3) $\Rightarrow$ (7) $\Rightarrow$ (8) and (2) $\Rightarrow$ (1) are trivial. □

Theorem 3.3. *Let $(X,\mathcal{T},\mathcal{T}^*)$ be a dfts. Then, the following statements are equivalent.*

(1) *$(X,\mathcal{T},\mathcal{T}^*)$ is a double fuzzy normal space (resp. double fuzzy almost normal space).*
(2) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\underline{1}-\lambda_2) \geq r, \mathcal{T}^*(\underline{1}-\lambda_2) \leq s$ and $\lambda_1\bar{q}\lambda_2$, there exist (r,s)-rfo sets μ_1,μ_2 such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1\bar{q}\mu_2$.*
(3) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\underline{1}-\lambda_2) \geq r, \mathcal{T}^*(\underline{1}-\lambda_2) \leq s$ and $\lambda_1\bar{q}\lambda_2$, there exist (r,s)-gfo sets μ_1,μ_2 such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1\bar{q}\mu_2$.*
(4) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\underline{1}-\lambda_2) \geq r, \mathcal{T}^*(\underline{1}-\lambda_2) \leq s$ and $\lambda_1\bar{q}\lambda_2$, there exist (r,s)-*gfαo *sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1\bar{q}\mu_2$.*
(5) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\lambda_2) \geq r, \mathcal{T}^*(\lambda_2) \leq s$ and $\lambda_1 \leq \lambda_2$, there exists $\rho \in I^X$ such that $\mathcal{T}(\rho) \geq r, \mathcal{T}^*(\rho) \leq s$ and*

$$\lambda_1 \leq \rho \leq C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \leq \lambda_2.$$

(6) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\lambda_2) \geq r, \mathcal{T}^*(\lambda_2) \leq s$ and $\lambda_1 \leq \lambda_2$, there exists an (r,s)-rfo set $\rho \in I^X$ such that*

$$\lambda_1 \leq \rho \leq C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \leq \lambda_2.$$

(7) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\lambda_2) \geq r, \mathcal{T}^*(\lambda_2) \leq s$ and $\lambda_1 \leq \lambda_2$, there exists an (r,s)-gfo set $\rho \in I^X$ such that*

$$\lambda_1 \leq \rho \leq C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \leq \lambda_2.$$

(8) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\lambda_2) \geq r, \mathcal{T}^*(\lambda_2) \leq s$ and $\lambda_1 \leq \lambda_2$, there exists an (r,s)-*gfαo *set $\rho \in I^X$ such that*

$$\lambda_1 \leq \rho \leq \alpha C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \leq \lambda_2.$$

(9) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\lambda_2) \geq r, \mathcal{T}^*(\lambda_2) \leq s$ and $\lambda_1 \leq \lambda_2$, there exists an (r,s)-*fαo *set $\rho \in I^X$ such that*

$$\lambda_1 \leq \rho \leq \alpha C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s) \leq \lambda_2.$$

(10) *For each $\lambda_1,\lambda_2 \in I^X$, $r \in I_0$ and $s \in I_1$ such that $\mathcal{T}(\underline{1}-\lambda_1) \geq r, \mathcal{T}^*(\underline{1}-\lambda_1) \leq s, \mathcal{T}(\underline{1}-\lambda_2) \geq r, \mathcal{T}^*(\underline{1}-\lambda_2) \leq s$ and $\lambda_1\bar{q}\lambda_2$, there exist (r,s)-*fαo *sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1\bar{q}\mu_2$.*

Proof. It is clear from Theorem 3.2. □

Theorem 3.4. *Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be dfts's such that $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ is a double fuzzy normal space. If $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \to (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is a double fuzzy almost continuous, double fuzzy almost closed and surjective function, then $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is a double fuzzy mildly normal space.*

Proof. For each $r \in I_0$ and $s \in I_1$, let $\nu_1,\nu_2 \in I^Y$ are (r,s)-rfc sets such that $\nu_1\bar{q}\nu_2$. Since f is double fuzzy almost closed function $\mathcal{T}_1(\underline{1}-f^{-1}(\nu_1)) \geq r, \mathcal{T}_1^*(\underline{1}-f^{-1}(\nu_1)) \leq s, \mathcal{T}_1(\underline{1}-f^{-1}(\nu_2)) \geq r, \mathcal{T}_1^*(\underline{1}-f^{-1}(\nu_2)) \leq s$ and $f^{-1}(\nu_1)\bar{q}f^{-1}(\nu_2)$. Since $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ is double fuzzy normal space, there exist $\mu_1,\mu_2 \in I^X$ such that $\mathcal{T}_1(\mu_1) \geq r, \mathcal{T}_1^*(\mu_1) \leq s, \mathcal{T}_1(\mu_2) \geq r, \mathcal{T}_1^*(\mu_2) \leq s$ and $f^{-1}(\nu_1) \leq \mu_1, f^{-1}(\nu_2) \leq \mu_2$ with $\mu_1\bar{q}\mu_2$. Since $I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu_1,r,s),r,s)$ and $I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu_2,r,s),r,s)$ are (r,s)-rfo sets and

$$I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu_1,r,s),r,s)\bar{q}I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu_2,r,s),r,s).$$

Furthermore,

$$f^{-1}(\nu_i) \leq \mu_i \leq I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu_i,r,s),r,s), \quad \text{for each } i \in \{1,2\}.$$

Since f is double fuzzy almost continuous, there exist $\gamma_1,\gamma_2 \in I^Y$ such that $\mathcal{T}_2(\gamma_1) \geq r, \mathcal{T}_2^*(\gamma_1) \leq s, \mathcal{T}_2(\gamma_2) \geq r, \mathcal{T}_2^*(\gamma_2) \leq s$ and $\nu_1 \leq \gamma_1, \nu_2 \leq \gamma_2$ with

$$f^{-1}(\gamma_i) \leq I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\mu_i,r,s),r,s), \quad \text{for each } i \in \{1,2\}.$$

Moreover $\gamma_1\bar{q}\gamma_2$. Hence $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is double fuzzy mildly normal space. □

Corollary 3.5. *Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be dfts's and let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ be double fuzzy normal space. If $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy almost continuous and double fuzzy closed function, then $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy mildly normal space.*

Corollary 3.6. *Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be dfts's and let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ be double fuzzy mildly normal space. If $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy almost continuous, double fuzzy almost closed and double fuzzy open (resp.double fuzzy continuous, double fuzzy closed, double fuzzy open) function, then $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy mildly normal space.*

Theorem 3.7. *Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be dfts's and let $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be double fuzzy mildly normal space (resp. double fuzzy normal space). If $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy almost regular generalized continuous, double fuzzy regular closed (resp. double fuzzy almost closed) injective function, then $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal space.*

Proof. For each $r \in I_0$ and $s \in I_1$, let $\lambda_1, \lambda_2 \in I^X$ are (r, s)-rfc sets such that $\lambda_1 \bar{q} \lambda_2$. Since f is double fuzzy regular closed (resp. double fuzzy almost closed) injective function, $f(\lambda_1)$ and $f(\lambda_2)$ are (r, s)-rfc sets (resp. $\mathcal{T}_2(\underline{1} - f(\lambda_1)) \geq r$, $\mathcal{T}_2^*(\underline{1} - f(\lambda_1)) \leq s$ and $\mathcal{T}_2(\underline{1} - f(\lambda_2)) \geq r$, $\mathcal{T}_2^*(\underline{1} - f(\lambda_2)) \leq s$). Since $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy mildly normal (resp. double fuzzy normal) space, there exist $\nu_1, \nu_2 \in I^Y$ such that $\mathcal{T}_2(\nu_1) \geq r$, $\mathcal{T}_2^*(\nu_1) \leq s$, $\mathcal{T}_2(\nu_2) \geq r$, $\mathcal{T}_2^*(\nu_2) \leq s$ and $f(\lambda_1) \leq \nu_1$, $f(\lambda_2) \leq \nu_2$ with $\nu_1 \bar{q} \nu_2$.

Now let $\gamma_1 = I_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\nu_1, r, s), r, s)$ and let $\gamma_2 = I_{\mathcal{T}_2,\mathcal{T}_2^*}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\nu_2, r, s), r, s)$. Then, γ_1, γ_2 are (r, s)-rfo sets such that $f(\lambda_1) \leq \gamma_1$, $f(\lambda_2) \leq \gamma_2$ and $\gamma_1 \bar{q} \gamma_2$. Since f is double fuzzy almost regular generalized continuous function, then $f^{-1}(\gamma_1)$ and $f^{-1}(\gamma_2)$ are (r, s)-rgfo sets. Furthermore, $\lambda_1 \leq f^{-1}(\gamma_1)$, $\lambda_2 \leq f^{-1}(\gamma_2)$ and $f^{-1}(\gamma_1) \bar{q} f^{-1}(\gamma_2)$. Hence by Theorem 3.2, $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal space. □

Theorem 3.8. *Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be double fuzzy regular continuous, double fuzzy almost generalized closed and surjective function. If $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal space, then $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy normal space.*

Proof. For each $r \in I_0$ and $s \in I_1$, let $\nu_1, \nu_2 \in I^Y$ such that $\mathcal{T}_2(\underline{1} - \nu_1) \geq r$, $\mathcal{T}_2^*(\underline{1} - \nu_1) \leq s$, $\mathcal{T}_2(\underline{1} - \nu_2) \geq r$, $\mathcal{T}_2^*(\underline{1} - \nu_2) \leq s$ and $\nu_1 \bar{q} \nu_2$. Since f is double fuzzy regular continuous, $f^{-1}(\nu_1)$ and $f^{-1}(\nu_2)$ are (r, s)-rfc with $f^{-1}(\nu_1) \bar{q} f^{-1}(\nu_2)$. Since $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal, there exist $\lambda_1, \lambda_2 \in I^X$ such that $\mathcal{T}_1(\lambda_1) \geq r$, $\mathcal{T}_1^*(\lambda_1) \leq s$, $\mathcal{T}_1(\lambda_2) \geq r$, $\mathcal{T}_1^*(\lambda_2) \leq s$ and $f^{-1}(\nu_1) \leq \lambda_1$, $f^{-1}(\nu_2) \leq \lambda_2$ with $\lambda_1 \bar{q} \lambda_2$.

Let $\mu_1 = I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\lambda_1, r, s), r, s)$ and let $\mu_2 = I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\lambda_2, r, s), r, s)$. Then, clearly μ_1 and μ_2 are (r, s)-rfo sets such that $f^{-1}(\nu_1) \leq \mu_1$, $f^{-1}(\nu_2) \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Since f is double fuzzy almost generalized closed, there exist $\gamma_1, \gamma_2 \in I^Y$ such that γ_1, γ_2 are (r, s)-gfo sets and $\nu_1 \leq \gamma_1$, $\nu_2 \leq \gamma_2$, $f^{-1}(\gamma_1) \leq \mu_1$ and $f^{-1}(\gamma_2) \leq \mu_2$. Since $\mu_1 \bar{q} \mu_2$, then $\gamma_1 \bar{q} \gamma_2$. But γ_1, γ_2 are (r, s)-gfo, $\nu_1 \leq I_{\mathcal{T}_2,\mathcal{T}_2}(\gamma_1, r, s)$ and $\nu_2 \leq I_{\mathcal{T}_2,\mathcal{T}_2^*}(\gamma_2, r, s)$. Furthermore $I_{\mathcal{T}_2,\mathcal{T}_2^*}(\gamma_1, r, s) \bar{q} I_{\mathcal{T}_2,\mathcal{T}_2^*}(\gamma_2, r, s)$. Hence $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy normal space. □

Corollary 3.9. *Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be double fuzzy regular continuous, double fuzzy closed and surjective function. If $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal space, then $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy normal.*

Theorem 3.10. *Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be double fuzzy regular irresolute (resp. double fuzzy almost continuous), double fuzzy almost regular generalized closed and surjective function. If $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal space (resp. double fuzzy normal space), then $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy mildly normal space.*

Proof. For each $r \in I_0$ and $s \in I_1$, let $\nu_1, \nu_2 \in I^Y$ are (r, s)-rfc sets such that $\nu_1 \bar{q} \nu_2$. Since f is double fuzzy regular irresolute (resp. fuzzy almost closed) function, $f^{-1}(\nu_1)$ and $f^{-1}(\nu_2)$ are (r, s)-rfc sets (resp. $\mathcal{T}_1(f^{-1}(\underline{1} - \nu_1)) \geq r$, $\mathcal{T}_1^*(\underline{1} - f^{-1}(\nu_1)) \leq s$, $\mathcal{T}_1(\underline{1} - f^{-1}(\nu_2)) \geq r$, $\mathcal{T}_1^*(\underline{1} - f^{-1}(\nu_2)) \leq s$) such that $f^{-1}(\nu_1) \bar{q} f^{-1}(\nu_2)$. Since $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy mildly normal space (resp. double fuzzy normal space), there exist $\lambda_1, \lambda_2 \in I^X$ such that $f^{-1}(\nu_1) \leq \lambda_1$, $f^{-1}(\nu_2) \leq \lambda_2$ and $\lambda_1 \bar{q} \lambda_2$.

Let $\mu_1 = I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\lambda_1, r, s), r, s)$ and let $\mu_2 = I_{\mathcal{T}_1,\mathcal{T}_1^*}(C_{\mathcal{T}_1,\mathcal{T}_1^*}(\lambda_2, r, s), r, s)$. Then, clearly μ_1, μ_2 are (r, s)-rfo sets such that $f^{-1}(\nu_1) \leq \mu_1$, $f^{-1}(\nu_2) \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Since f is double fuzzy almost regular generalized closed function, there exist $\gamma_1, \gamma_2 \in I^Y$ such that γ_1 and γ_2 are (r, s)-rgfo sets, $\nu_1 \leq \gamma_1$, $\nu_2 \leq \gamma_2$, $f^{-1}(\gamma_1) \leq \mu_1$ and $f^{-1}(\gamma_2) \leq \mu_2$. Since $\mu_1 \bar{q} \mu_2$, then $\gamma_1 \bar{q} \gamma_2$. Hence $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is fuzzy mildly normal space. □

Corollary 3.11. *Let $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be double fuzzy almost continuous, double fuzzy almost closed and surjective function. If $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ is double fuzzy normal space, then $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is double fuzzy mildly normal space.*

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